Continuously varying critical exponents in a sandpile model with internal disorder

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A sandpile model with an internal disorder is presented. The updating of critical sites is done according to a stochastic rule (with a probabilistic toppling q). Using a unified mean-field theory and numerical simulations, we have shown that the criticality is ensured for any value of q. The static critical exponents have been calculated and found to be the same as those obtained for the deterministic sandpile model, which is a particular case of the stochastic model. They have a universal q-independent behavior. In the limit of slow driving, we have developed a relation between our model and the branching process in order to compute the size exponent τ. It presents a continuous variation with the parameter of toppling q.

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I. INTRODUCTION

Piles of granular material are examples of driven open dynamical systems that often demonstrate catastrophic events in the form of avalanches. In a series of papers [1–4], Bak and co-workers introduced the concept of ''self-organized criticality'' (SOC) in order to study these systems. The basic idea is that these systems are driven into a critical state without a fine tuning of any control parameter. The SOC phenomenon was observed in other fields such as biology [5], economics [6], forest fire models [7], earthquakes [8], the game of life [9], invasion percolation [10], etc. It is often asserted that the sandpile models might explain the frequent occurrence of both $1/f^{\alpha}$ noise [11] and self-similar spatial structures. Therefore, the understanding of the self-organized critical system behavior promised to bring more light into the creation of the $1/f^{\alpha}$ type noise signals and fractal structures in nature.

Such models exhibit different behaviors depending on whether the rules of evolution are based on the absolute sand heights [12,13] of the pile or whether they depend on the local slopes [12–14] or on the Laplacian of the sand height function [12]. So as to explain this new phenomena, some theories have been elaborated. Dhar showed that a large class of absolute height was Abelien [15]. This property led to a particularly simple equiprobable partitioning in configuration space and allowed an exact solution of these models. A sandpile model with anisotropic geometry was studied within a dynamical renormalization group framework and various critical exponents were determined exactly in all dimensions [16]. Nevertheless, the universality classes in sandpile models remains an unresolved issue [17,18]. Even if it neglects correlations, the mean-field theory remains the simplest theoretical approach to SOC, which gives qualitative and quantitative descriptions of the system behavior [2,19–26].

Although SOC is frequently observed in sandpile models using numerical simulations [1,12,27,28] or analytical approaches [2,19,25], there is no evidence of self-organized critical dynamics in real piles of granular material. Frette *et al.* [29] studied experimentally the dynamical behavior of the driven quasi-one-dimensional pile of rice. They showed that the dynamic depends on the grain shape. The SOC state, in which the avalanche size distribution has a power-law component with critical exponent $\tau=2.02$, has been established for elongated grains. The experimental results of Frette *et al.* [29] and other theoretical studies of the dynamics of granular material [30] reveal that many physical properties of the pile granular material are important for the SOC state to be established. Several theoretical efforts have been made so as to describe real piles of granular material by incorporating randomness into the toppling rules [18,31–34]. However, the gravity effects, grain friction, and the local conditions on the pile are taken into account in a simple way, through a parameter p. The value of the parameter p decides whether the grain will stop on the site or topple to neighboring sites.

Disorder in sandpile models can be induced from probabilistic rules. Lübeck *et al.* [34] studied numerically a sandpile model which exhibits self-organized criticality and nonequilibrium phase transition as the probabilistic toppling parameter p is varied. When the system displays SOC behavior, the corresponding exponents have a nonuniversal p-dependent behavior. This nonuniversal behavior was observed also in a class of nonlocal unlimited sandpile models with stochastic dynamics [32]. Then, the two critical exponents which determine the dependence of the probability density on time and system size both depend continuously on a probability toppling parameter. A stochastic dynamics was elaborated following a different method by introducing a temperaturelike parameter T [35,36]. As a matter of fact, the criticality of sandpiles in thermal equilibrium is ensured with apparently continuously varying critical exponents at $T>0$. On the other hand, the friction effect between the grains in a pile of granular material was modeled by introducing a stochastic rule in the relaxation process [33]. Such models with local and nonlocal relaxations belong to different universality classes. Recently, varying critical exponents were observed in an artificial version of the Dhar and Ramaswamy sandpile model [37].

The aim of this paper is to investigate the stochastic effect of critical sites in a sandpile model using a unified mean-field theory proposed by Vespignani and Zapperi [25] and rejoin the branching process limit [21] in order to compute the dynamical critical exponents. Our model contains three

types of sites: stable, critical, and active, and we assume that the critical sites possess a probabilistic toppling parameter q. We believe that the introduction of such a parameter takes into account the microscopic details of the process and then describes an internal randomness, which is inherent in the dynamics. We show that the stochasticity does not affect either the phase diagram or the static critical exponents of deterministic sandpiles, while the size distribution critical exponent τ depends strongly on the parameter q. In the following section we will describe our model. In Sec. III we develop the method and illustrate the results. Finally, we conclude in Sec. IV.

II. MODEL

Recently, a great deal of effort has been made in the study of SOC introduced by Bak and co-workers. In this new phenomena, a system, under time evolution of its own dynamics, reaches a critical state which lacks any characteristic length or time scales and obeys power-law distributions.

Sandpile models are cellular automata defined on a d-dimensional lattice. They are characterized by an integer variable (energy) z_i at all sites i. The system is then driven by choosing a random site i and incrementing the value of z_i by 1, i.e.,

$$z_i \rightarrow z_i + 1. \tag{1}$$

If z_i thereby exceeds a threshold value z_c, the site i topples and the grains are distributed from the site i to its nearest neighbors (NN), i.e.,

$$z_i \rightarrow z_i - z_c, \tag{2}$$

$$z_{NN} \rightarrow z_{NN} + 1,$$

possibly causing some of them to be unstable too, and a cascade of toppling or avalanche is triggered. The avalanche may continue for some time and is stopped only if all variables z_i are below the threshold value or by sand falling off the edges. The total amount of sliding, S, induced by the single perturbation represents the avalanche size. After each perturbation, the stationary state is restored and another site is perturbed and so on. Therefore, the dynamics of the model summarizes the deposition and relaxation and it is assumed that the relaxation process is considered to be fast compared to the deposition. The separation between the buildup and the relaxation has profound effects on the global avalanche statistics. Under this condition, the system reaches a stationary state characterized by avalanches whose sizes S are distributed as a power law [1,12,14,38].

As a basic model we use that proposed by Vespignani and Zapperi [25], and investigate the robustness of the critical behavior upon the modification of the toppling process. Specifically, we study a stochastic variant of the model proposed in Ref. [25]. Then, in our model each site can be characterized by three possible states: active, when $z > z_c$; critical, when $z = z_c$; and stable for $z < z_c$. We assume that the critical state can topple at time t with a probability q if one of its NN topples at a preceding step $t-1$, while the active site relaxes immediately as it receives a grain of energy. The updating of toppling sites is in parallel. In contrast to Ref. [25], a critical site has a stochastic character. The avalanche stops at a time t as soon as all sites i have $z < z_c$ and no toppling occurs due to the internal disorder. The energy gain given by the presence of internal randomness can be used only if the site is in the front of an avalanche, i.e., if the avalanche touches one of its neighbors. We are accustomed, using numerical simulations, to separating time scales: any avalanche that might be started by a deposited grain will have ended before a new grain is deposited. However, the site evolution depends upon the whole system configuration and it is hard to describe this nonlocal interactions. Thus, we incorporate all these interactions in a nonvanishing probability h. By formal analogy with critical phenomena, the role of the magnetic field h is played by the external flux of sand, i.e., h is the probability per unit time that a site will receive a grain of energy. Therefore, the state of a single site depends only on the state of the site itself and its NN sites at previous time step via the transition probabilities.

III. METHOD AND RESULTS

A. Mean-field theory

As in Ref. [25] the simplest description of SOC models is through a d-dimensional stochastic cellular automata with $N = L^d$ sites, where L is the lattice linear size. Each site i on the lattice is characterized by an occupation variable s_i, which can assume different values depending on the energy accumulated in this site. The whole lattice is then characterized by a configuration $s = \{s_i\}$ and the evolution of the system is determined by the transition probability $W(s/s')$ from configuration s' to configuration s. At each time step the state of a given site depends only on the previous state of the site itself and the set of sites interacting with it. Thus, we consider the probability distribution $P(s,t)$ to have a configuration s at time t. Then in a configuration s and at time t we can express the average value of any function $A(s,t)$ by

$$\langle A(t) \rangle = \sum_s A(s,t) P(s,t). \tag{3}$$

The dynamical evolution of the probability distribution is given by the following continuum master equation (ME):

$$\frac{\partial}{\partial t} P(s,t) = \sum_{s'} W(s/s') P(s',t) - W(s'/s) P(s,t), \tag{4}$$

where the first term of the right hand side expresses the input rate from the configuration s and the second term expresses the output rate from the configuration s to the configuration s'. For nonequilibrium systems we must solve the ME in the stationary state limit. To do this, we use the dynamic cluster variation approach [22,25] within the mean-field theory, which neglects the correlations up to a certain order.

Before proceeding to give different transitions probabilities between several configurations, we first develop some ideas concerning the dynamical evolution of our system. As has been underlined in the preceding section, we can assume

mainly for each site three states: active ($s_i=a$), critical ($s_i=c$), and stable ($s_i=s$). Active sites are those transferring energy by interacting with nearest neighbors; critical sites become active with the addition of energy; and stable are those that do not relax if energy is added to them by external fields or interactions with active sites. This description is only approximate, since a certain amount of information is lost in grouping together stable sites. In the simple mean-field single-site approximation, we denote by ρ_a, ρ_c, and ρ_s, the average densities of sites in the active, critical, and stable states, respectively. These densities can be written as

$$\rho_k(t)=\sum_{\{s\}}\delta(s_i-k)P(s,t), \tag{5}$$

where $k=a,c,s$ and δ denotes the Kronecker symbol. Finally, we get [32]

$$\frac{\partial}{\partial t}\rho_k(t)=\sum_{\{s'\}}w(k/s')\prod_i\rho_{s_i'}-\rho_k(t), \tag{6}$$

where $w(k/s')$ is the one-site transition probability and $\rho_{s_i}(t)$ measures the single-site probability. All the dynamical information of the system is contained in the transition rates $w(k/s')$ given by the reaction rates. However, the sandpile model is inherently nonlocal. Thus, as it has been evoked in the preceding section, by introducing the external flow of energy we recuperate the nonlocal interactions. Consequently, the transition rates depend only on the field h and on the state of the NN sites that determine the toppling dynamics and then approximate an avalanche. The nonlocality of the dynamical rules is recovered in the limit $h\to 0$ (see Sec. III B).

Independently of the initial conditions, the system organizes itself in a critical state characterized by a power-law distribution and therefore without any characteristic length or time scale. The interesting point is that unlike ordinary critical phenomenon, no tuning of any control parameters is necessary to reach this state. In other words, the critical state is an attractor for the dynamics, and the parameters of the system flow spontaneously to the critical value. For this reason, it is interesting to investigate how robust the critical steady states of these sandpile models are under changing the dynamical rules which define them. In our model we introduce internal perturbation, which is related to the critical sites; so if the energy of site i is equal to the threshold value z_c and if one of its NN topples at a preceding step $t-1$, the toppling in the site i will occur only with a probability q. Thus, our model differs from that proposed by Vespignani and Zapperi [25] because the randomness is internal and intrinsic in the dynamics.

Now we can write the transition probabilities by neglecting the contribution due to the presence of multiple active NN sites. The density of active sites is derived by considering all contributions: the allowed transitions to the active state are due to the noncritical toppling sites that can receive energy from the external field or from the NN sites. The first possibility to consider is that all the NN sites are stable and the addition of energy comes from outside, so we have

$$\sum_{\{s'_{NN}\}}w(a/c,s'_{NN}\neq a \text{ and } \neq c_t)(1-qA_c)\rho_c\prod_{i\in NN}\rho_{s_i'(t)}$$
$$=h(1-qA_c)\rho_c(1-\rho_a-qA_c\rho_c)^Z, \tag{7}$$

where z represents the number of NN sites, and the factor A_c denotes the probability that one of the neighboring sites is active at a previous time. The notation c_t refers to the critical sites which can topple with a probability q if one of its nearest neighbors topples at a previous step.

The possibilities other than the first one are the result of the transfer of energy from NN sites. This process corresponds to the following two terms:

$$\sum_{\{s'_{NN}\}}w(a/c,s_i'=a,s'_{j\neq i}\neq a \text{ and } \neq c_t)\times(1-qA_c)\rho_c\rho_a\prod_{j\neq i_{NN}}\rho_{s_j'(t)}$$
$$=(1-h)(g-1-\epsilon)(1-qA_c)\rho_c\rho_a\times(1-\rho_a-qA_c\rho_c)^{Z-1}, \tag{8}$$

$$\sum_{\{s'_{NN}\}}w(a/c,s_i'=c_t,s'_{j\neq i}\neq a \text{ and } \neq c_t)\times(1-qA_c)\rho_c qA_c\rho_c\prod_{j\neq i_{NN}}\rho_{s_j'(t)}$$
$$=(1-h)(g-1-\epsilon)(1-qA_c)\rho_c qA_c\rho_c\times(1-\rho_a-qA_c\rho_c)^{Z-1}, \tag{9}$$

where the parameter ϵ identifies the average energy dissipated in each elementary process, which can be due to the boundary or bulk dissipation [25].

Equation (8) expresses the fact that only one of the NN sites is active and yields its energy to the site i, the other NN sites are neither active nor critical toppling sites; while in Equ. (9) only one of the NN sites is a critical toppling site which transfers its energy to the site i. We suppose that the active site ($s=a$) and the critical toppling site ($s=c_t$), which contain g and $(g-1)$ grains of energy, respectively, by relaxing lose $(g-1)$ grains of energy. Thus the active site that relaxes creates a stable site with one level of energy, and the critical toppling site generates a stable site with a zero level of energy. Neglecting higher orders in h and ρ_a from Eqs. (8) and (9), we can finally write the mean-field (MF) dynamical equation for the densities of active sites:

$$\frac{\partial\rho_a}{\partial t}=-\rho_a+(g-1-\epsilon)\rho_c(\rho_a+qA_c\rho_c)(1-qA_c)$$
$$+h\rho_c(1-qA_c). \tag{10}$$

Next, we derive the dynamical MF equation for the density of stable sites. The density of critical sites can be deduced from the normalization equation. As it has been emphasized above, the stable state contains several levels of energy. In fact,

$$\rho_s=\sum_{n=0}^{g-2}\rho_n,$$

where ρ_n describes the probability that a site is in level n. Thus, following the same strategy used above, we get

$$\frac{\partial \rho_0}{\partial t} = -h\rho_0 - (g-1-\epsilon)\rho_0(\rho_a + qA_c\rho_c) + qA_c\rho_c, \tag{11}$$

$$\frac{\partial \rho_1}{\partial t} = -h\rho_1 - (g-1-\epsilon)\rho_1(\rho_a + qA_c\rho_c) + \rho_a + h\rho_0 + (g-1-\epsilon)\rho_0(\rho_a + qA_c\rho_c), \tag{12}$$

and for $2 \leqslant n \leqslant g-2$,

$$\frac{\partial \rho_n}{\partial t} = -h\rho_n - (g-1-\epsilon)\rho_n(\rho_a + qA_c\rho_c) + h\rho_{n-1} + (g-1-\epsilon)\rho_{n-1}(\rho_a + qA_c\rho_c). \tag{13}$$

So we have

$$\frac{\partial \rho_s}{\partial t} = -h\rho_{g-2} - (g-1-\epsilon)\rho_{g-2}(\rho_a + qA_c\rho_c) + \rho_a + qA_c\rho_c. \tag{14}$$

If we denote by u the fraction of stable sites receiving a quantum energy that will contribute to the $s \to c$ process, i.e., the fraction of stable sites which are subcritical,

$$u = \frac{\rho_{g-2}}{\rho_s} \sim \frac{\rho_{g-2}}{\sum_{i=0}^{g-2} \rho_i}, \tag{15}$$

the dynamical equation for density of stable sites is then

$$\frac{\partial \rho_s}{\partial t} = -uh\rho_s - (g-1-\epsilon)u\rho_s(\rho_a + qA_c\rho_c) + \rho_a + qA_c\rho_c. \tag{16}$$

If we consider all the possible processes for A_c, we can write

$$A_c = \rho_a(1-\rho_a)^{z-1}\rho_{g-2} \approx \rho_a\rho_{g-2}, \tag{17}$$

$$A_c = u\rho_s\rho_a. \tag{18}$$

In the stationary state, we have $\rho_1 = \rho_2 = \cdots = \rho_{g-2}$. Then, we obtain the result

$$u = \frac{1}{g-1}. \tag{19}$$

In the following, we will consider the stationary properties. Combining the normalization equation with the stationary limit of Eqs. (10) and (14), we obtain the following set of coupled equations:

$$\rho_a = uh\rho_s + u(g-1-\epsilon)\rho_s(\rho_a + uq\rho_s\rho_a\rho_c) - uq\rho_s\rho_c\rho_a,$$

$$\rho_a = h\rho_c(1-uq\rho_s\rho_a) + (g-1-\epsilon)\rho_c(\rho_a + uq\rho_s\rho_a\rho_c) \times (1-uq\rho_s\rho_a), \tag{20}$$

$$\rho_a = 1 - \rho_s - \rho_c.$$

By expanding $\rho_a(h)$ for small values of h and after some algebra, we obtain from Eqs. (20),

$$\rho_a(h) = \frac{h}{(1+qu\rho_s\rho_c)\epsilon} + \theta(h^2). \tag{21}$$

By substituting the expression of the active density ρ_a in the second equation of the system (20) we obtain the following equation, which connects only the stable and critical densities:

$$(g-1)\rho_c + uq(g-1)\rho_s\rho_c^2 - 1 = 0, \tag{22}$$

or in the stationary state no active site is present in the system. Then, using the normalization equation $\rho_s \sim 1-\rho_c$ we obtain a closed equation

$$uq\rho_c^3 - uq\rho_c^2 - \rho_c + \frac{1}{g-1} = 0, \tag{23}$$

for which the solution can be approached by a polynomial function on q:

$$\rho_c = \frac{1}{g-1} - qf_1(1/g) + q^2 f_2(1/g) + \theta(q^3), \tag{24}$$

where $f_i(1/g)(i=1,2)$ are polynomial functions on $1/g$. In analogy with other nonequilibrium phenomena [39,40], the order parameter is defined by $P_a = \rho_a + qu\rho_s\rho_c\rho_a$, which vanishes at the critical point. In order to define the response function that determines how the system rearranges itself, we study the effect of infinitesimal perturbation Δh on the steady state:

$$\chi_{h,\epsilon} = \lim_{h\to 0} \frac{\Delta P_a}{\Delta h} = \left.\frac{\partial P_a}{\partial h}\right|_{h=0}. \tag{25}$$

Since the density of active sites can be written as

$$\rho_a(h) = \frac{h}{(1+qu\rho_s\rho_c)\epsilon} + \theta(h^2), \tag{26}$$

the singular part of the susceptibility diverges as

$$\chi \propto \epsilon^{-\gamma} \quad \text{with} \quad \gamma = 1. \tag{27}$$

Then, a long-ranged response function occurs for $\epsilon = 0$, where the system reaches its critical point. This behavior is confirmed by studying numerically, for different values of q (Fig. 1), the response of a bidimensional system of linear size $L=64$ under infinitesimal h perturbation, which is the probability for unit time that a site receives an energy grain. We model the dissipation by introducing a probability $p = \epsilon/(g-1)$, for which the energy in relaxation is lost rather then being transferred. We observe that the susceptibility diverges at $\epsilon = 0$ for any value of q. As the driving h and the dissipation ϵ rates identify the two relevant scaling fields, the phase diagram we obtain is similar that of deterministic

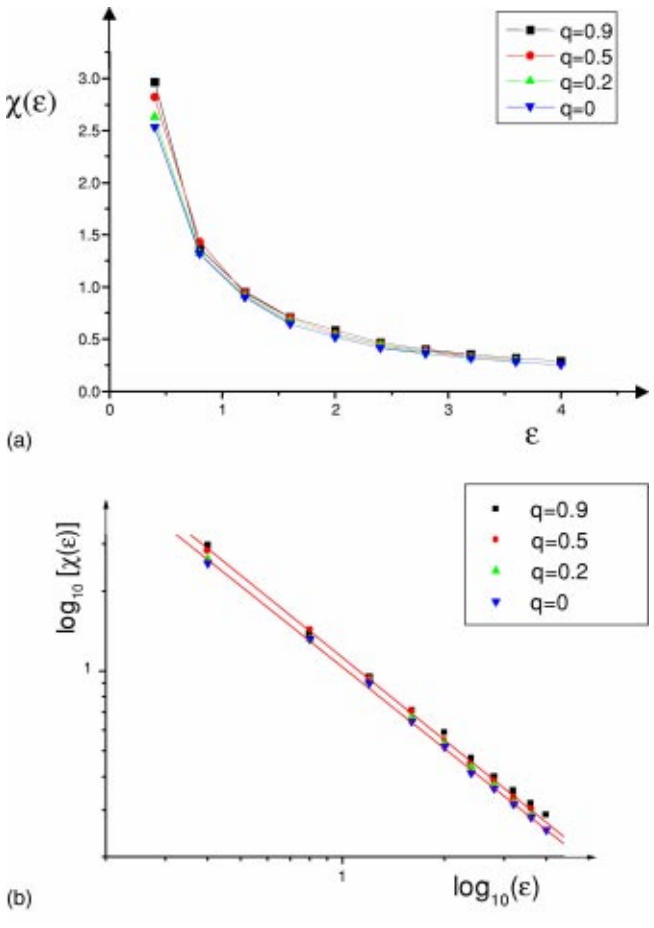

FIG. 1. (a) The susceptibility $\chi_\epsilon = dP_a/dh$ as a function of a dissipation ϵ with different values of q for a system with periodic boundary conditions and size $L=64$. (b)The $\log_{10}$-$\log_{10}$ plot of the susceptibility versus q. The straight lines represent the fit of two values of q ($q=0$ and 0.5). The mean-field critical exponent γ associated with the susceptibility is q-independent.

sandpiles [25], namely, that the model is supercritical for $h>0$, while for $h=0$ and $\epsilon>0$, it is subcritical and the dynamics displays avalanches. In spite of the internal disorder induced by stochastic rules, the mean-field critical exponents γ, μ, ν, and δ associated, respectively, with the susceptibility χ, $\chi \sim \epsilon^{-\gamma}$; the dissipation ϵ, $\epsilon \sim L^{-\mu}$; the correlation length ξ, $\chi \sim \xi^{-\nu}$; and the order parameter P_a, $P_a \sim h^{1/\delta}$ are the same as those of the original model [25] ($\gamma=1$, $\mu=2$, $\nu=1/2$, $\delta=1$). These results agree with those obtained by Caldarelli [41], who introduced in the toppling rules a probability that depends on a parameterlike temperature. We have studied numerically the model introduced by Bak, Tang, and Weisenfeld (BTW) with open boundaries and a finite driving rate h for a bidimensional system. The dissipation is considered only through the open boundary where the site transfers its energy outside. The variation of the order parameter versus h (Fig. 2) is linear with a slope that increases with the system size as L^2 and a density of active sites that goes to zero in the $h \to 0$ limit ($\delta=1$) independently of q (Fig. 3), which supports the result given by the mean-field theory.

Moreover, the ''cluster variational'' method takes into account the long-term behavior of the system and we can approximate the avalanches by several series of successive topples. Thus, we prepare the system in one of the ''natural'' configurations corresponding to the limit of infinitesimal driving, $\Delta h(x) = \delta(t)\delta(x)$, i.e., consisting of a single active site. The perturbation decays in the stationary subcritical state as

$$\rho_a(t) \sim t^{\eta} F(t/t_c(\epsilon)), \tag{28}$$

where we introduce the power-law exponent η and the cutoff characteristic time t_c. For a small perturbation close to the stationary state, we consider $\rho_k(t) = \rho_k + \delta\rho_k(t)$, where $\delta\rho_k(t)$ is the deviation of the densities from their stationary value. If we recall the expression of the order parameter P_a and rewrite its evolution equation

$$\frac{\partial}{\partial t} P_a = (1 + uq\rho_s\rho_c)[-\rho_a + (g-1-\epsilon)\rho_c(\rho_a + qA_c\rho_c)], \tag{29}$$

we will obtain

$$\frac{\partial}{\partial t}(\delta P_a) = (1 + uq\rho_s\rho_c)[-\delta\rho_a + (g-1-\epsilon)\rho_c\delta\rho_a \times (1 + qu\rho_c\rho_s)]. \tag{30}$$

Then, the linearized dynamical equation in diagonal form is given by

$$\frac{\partial}{\partial t}(\delta P_a) = (\delta P_a)\left(-\frac{\epsilon}{g-1}\right), \tag{31}$$

for which the solution is

$$\delta P_a(t) \sim \exp\left(\frac{-\epsilon t}{g-1}\right), \tag{32}$$

which implies $\eta=0$. Since the relaxation behavior follows an exponential law

$$\delta P_a(t) \sim \exp(-t/t_c), \tag{33}$$

the characteristic relaxation time depends on the toppling probability q and its behavior is given by

$$t_c \sim 1/\epsilon. \tag{34}$$

Since our model is based on stochastic rules, its dynamics is different from that of the deterministic sandpile model. Both active sites and critical sites which topple with a probability q contribute to the avalanche. We are therefore enable to derive scaling relations that allow us to compute the dynamical exponents as was done in Ref. [25]. Then, the distribution size critical exponent τ is calculated using the relation between the formalism presented above and the branching processes.

B. Branching process and numerical simulations

The spreading of an avalanche in mean-field theory can be described by a front consisting of noninteracting particles

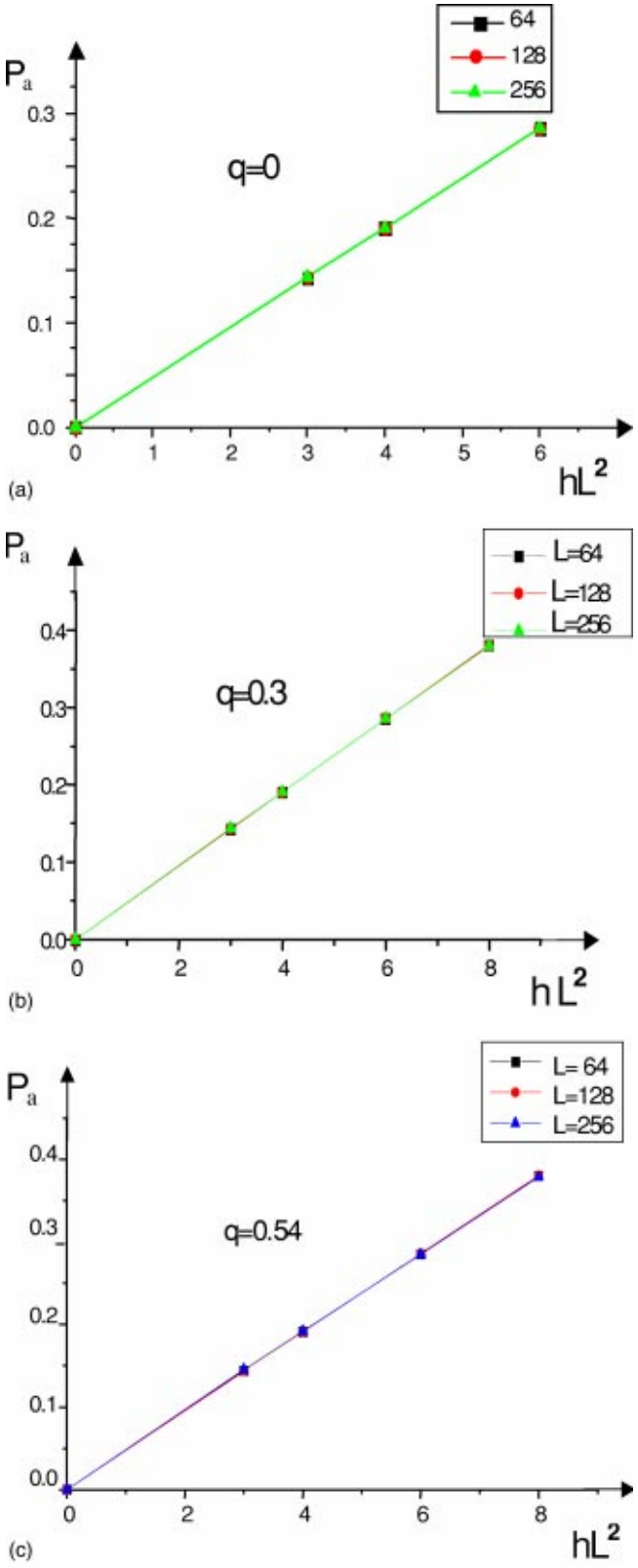

FIG. 2. The order parameter as a function of the driving rate h for different values of q, (a) $q=0$, (b) $q=0.3$, (c) $q=0.54$ in the BTW model with boundary dissipation plotted for different system sizes L.

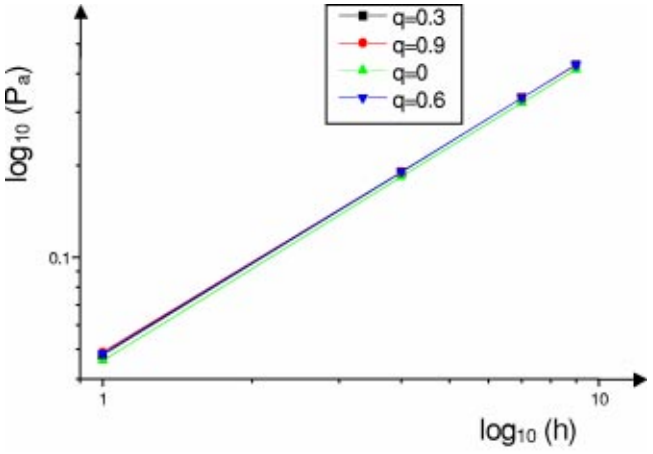

FIG. 3. $\log_{10}$-$\log_{10}$ plot of the order parameter P_a versus h with $\epsilon=0.01, g=3$, and system size $L=128$. We show that the critical exponent δ is constant with the variation of q.

that can either trigger subsequent activity or die out. This kind of process is well known and is called the branching process [42]. The SOC has been investigated and it has been proposed that the mean-field behavior of sandpile models can be described by a critical branching process [19–21,23,43]. A branching process [42] is defined by the number of active sites that can either die or generate n new sites with certain probabilities. The case $n=2$ is the simpler example: a site relaxes with probability p, leading to two new active sites, and it dies with probability $(1-p)$. There is a critical value $p_c=1/2$ such that for $p>p_c$ the probability to have an infinite avalanche is nonzero, while for $p<p_c$ all avalanches are finite. Thus $p=p_c$ corresponds to the critical case, where avalanches are power-law distributed [21]. Thus, the avalanche can be described as a branching process with an effective parameter that depends on the detail of the model under study.

In our model, the branching process is associated with the propagation of active and toppling critical sites in the subcritical regime. In the stationary state for $h=0$, an active site and critical toppling site generates $k=1,\ldots,g-1$ new active and critical toppling sites with probabilities

$$P_k=(1-\epsilon)C_{g-1}^{k}(\rho_c+uq\rho_s)^k(1-\rho_c-uq\rho_s)^{g-1-k}, \tag{35}$$

while no active or toppling critical sites are generated with probability

$$P_0=\epsilon+(1-\epsilon)(1-\rho_c-uq\rho_s)^{g-1}. \tag{36}$$

Then, the branching process control parameter for our model is given by

$$\tilde{P}=\frac{1}{g-1}\sum_k kP_k=(1-\epsilon)(\rho_c+uq\rho_s), \tag{37}$$

with a critical value $\tilde{P}_c=\rho_c^*+uq\rho_s^*$, where ρ_s^* and ρ_c^*, are respectively, the stationary values of critical and stable den-

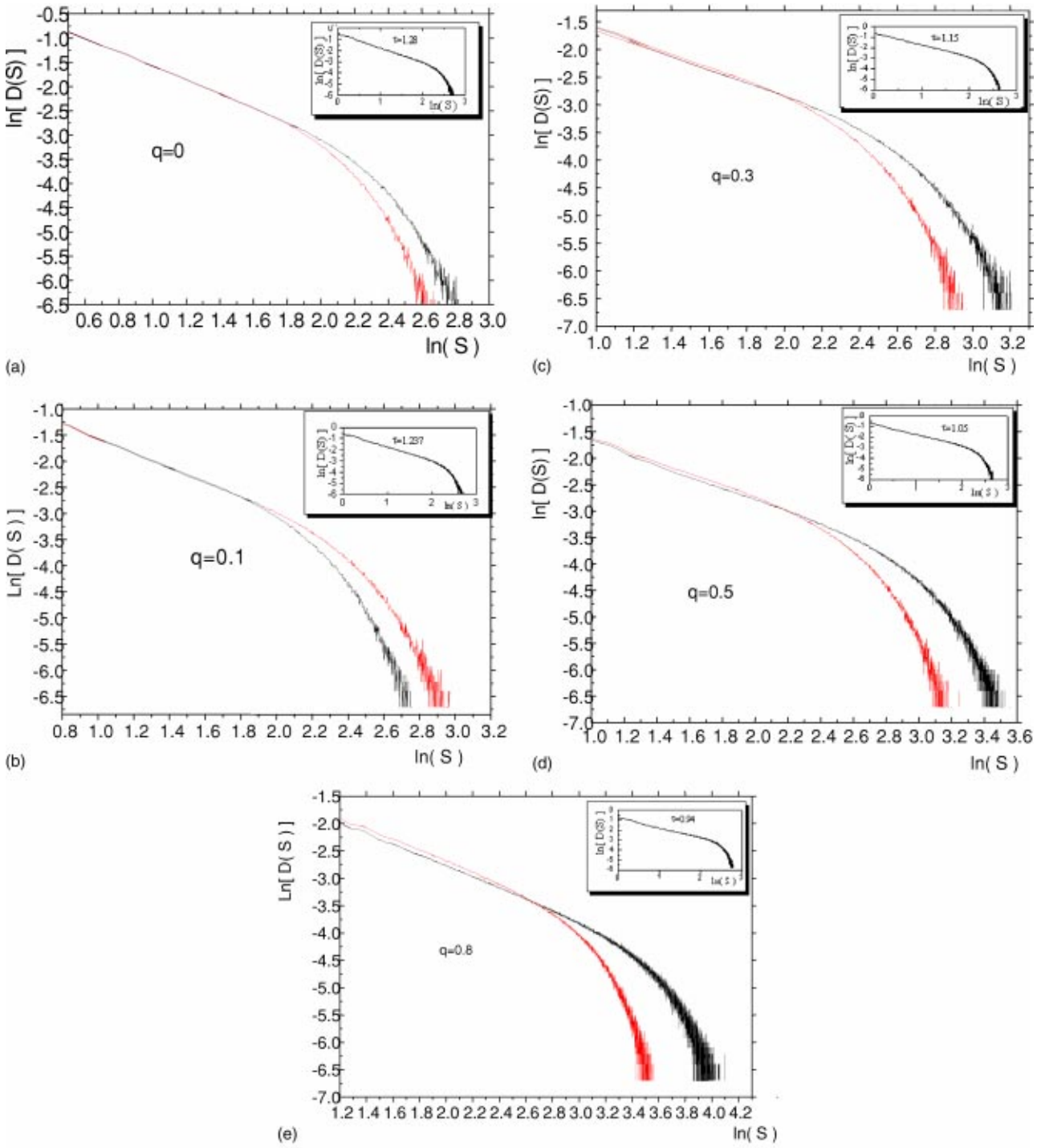

FIG. 4. The size distribution $D(S)$ for two generations $n=16$ and $n=20$ with $g=3$. The dynamical exponent varies continuously with q. We obtain for different values of q ranging from 0 to 0.8, respectively, $\tau=1.5$, 1.37, 1.19, 1.07, and 0.98. The insets in these figures show the numerical simulation of the BTW model with system size $L=32$; the dynamical exponent takes, respectively, the values $\tau=1.28$, 1.23, 1.14, 1.05, and 0.94 for $q=0$, 0.1, 0.3, 0.5, and 0.8.

sities. For $q=0$ we recover $\tilde{P}_c=1/2$, which is in agreement with Ref. [21]. In our analysis, we focus on the model properties in the critical steady state characterized by $\tilde{P}_c$. We study the model by carrying out simulations for different system sizes with $g=3$, and averaging over typically 5 $\times 10^6$ realizations. The model leads to a variety of states with widely varying avalanche sizes. We define the size of an avalanche as the number of topplings S, and investigate the distribution of S. In Fig. 4, we have shown for 16 and 20 generations, the avalanche size distribution for different val-

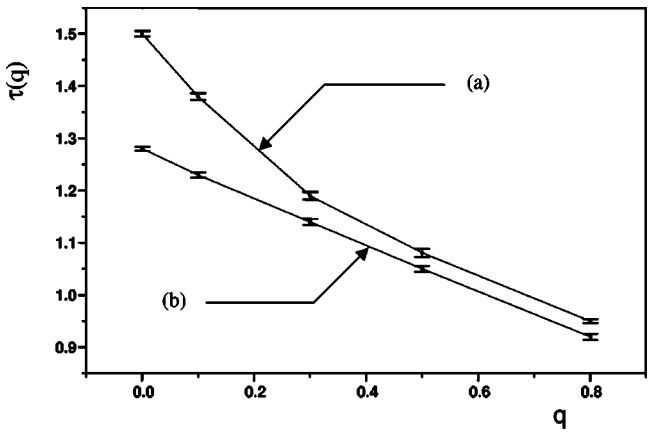

FIG. 5. Dynamical exponent versus q obtained (a) by the branching processes and (b) by numerical simulation of the BTW model.

ues of the toppling parameter q. For $q=0$ we observe a scaling region $D(S)\sim S^{-\tau}$ with $\tau=3/2$. Then, we find again the value given by both the mean-field theory and branching process for a deterministic sandpile model. By varying the parameter q, the critical state is ensured, but the value of the critical exponent τ decreases continuously by increasing the value of q. This result is consistent with Ref. [32]. In Fig. 4. we have shown the size distribution for $q=0.1$, 0.3, 0.5, and 0.8, which lead, respectively, to the critical exponents, $\tau=1.37$, 1.19, 1.07, and 0.98. In the limit $q\rightarrow 1$, the model is deterministic but not equivalent to the model presented by Vespignani and Zapperi [25]. Indeed, the dynamics is rather different, since only critical sites that receive a grain of energy at a previous time topple with a probability $q=1$. As a result, the model in the limit $q\rightarrow 1$, belongs to another universality class. To support the result obtained using the branching process, we have performed numerical simulations for a BTW model on a square lattice of size $L=32$. In Fig. 4, we have shown that the numerical results agree with those obtained by mean-field theory, namely, that the dynamical critical exponent τ decreases continuously by varying the toppling probability q. We remark that the mean-field values of τ are greater than numerical ones, and they get closer as $q\rightarrow 1$ (Fig. 5). In addition, we note that the dynamical critical exponents vary linearly within numerical simulations.

IV. CONCLUSION

To summarize, a sandpile model with a stochastic dynamics is studied using a single-site approximation to the master equation obtained from a mean-field theory. Within this approach, and by computing the order parameter, we have presented the mean effect of the stochastic rule with probabilistic toppling parameter q. In the limit of infinitesimal driving h, the system is subcritical for $\epsilon>0$ and displays a critical behavior. The phase diagram obtained from the control parameters is similar to those of deterministic sandpile models. The static critical exponents (γ,μ,ν,δ) are independent of the internal disorder, i.e., q, and they keep the same values, within the mean-field theory, obtained for the sandpile model with deterministic rules. In order to support the mean-field results, we have studied numerically the BTW model in $d=2$ in both cases: with a fixed dissipation ϵ and periodic boundary conditions so as to study the behavior of the susceptibility χ, and with a finite driving rate h and boundary dissipation for open boundary conditions, in order to show that the order parameter vanishes linearly with $h(\delta=1)$. As a result, the predictions of the mean-field theory seem to be correct and valid for any dimensions independently of the internal disorder. Since the dynamics of our model evolves within stochastic rules, we are not able to establish scaling relations which allow us to compute the dynamical exponents. However, using the branching process we have defined an adequate control parameter for our model and generate avalanches of different sizes. Then, we have shown that the size distribution critical exponent decreases continuously by increasing the value of q. Therefore, the dynamical exponents have a nonuniversal behavior, i.e., q-dependent behavior, in agreement with other stochastic sandpile models [34]. This result has been confirmed using numerical simulations. Since the metastability in sandpile models can result from competition between friction and gravity effects, we think that varying critical exponents are the result of stochastic dynamics which model metastable sites. However, it is worthwhile to confirm analytically this result.

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